Application of next-generation reservoir computing for predicting chaotic systems from partial observations

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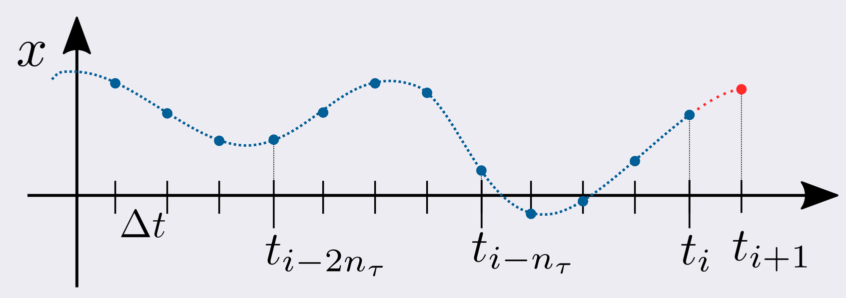
Introduction

• Next-generation reservoir computing (NG-RC) is a machine learning approach recently proposed as an effective method for predicting the dynamics of chaotic systems [1]. It uses a nonlinear vector autoregression (NVAR) algorithm with a feature vector consisting of several time-delayed input signals and their nonlinear functions usually represented in monomial form.

• Here we study the effectiveness of the NG-RC method when only a scalar time series is available for observation. We found that the prediction is only effective if the feature vector of the NVAR algorithm contains monomials of sufficiently high degree. To improve the prediction, we propose a modified algorithm, called NG-RC-Ch, in which monomials are replaced by Chebyshev polynomials of the first kind [2].

Theory

Suppose we have a discrete time series of data $\{x_1, x_2, \ldots\}$ sampled at regular time steps from a scalar observable $x(t)$ of a chaotic system, $x_i = x(i\Delta t)$:



The multidimensional state space is reconstructed by using the time delay coordinates. The embedding dimension k is estimated using the false nearest neighbors algorithm, and the delay time $\tau = n_\tau \Delta t$ – by the first minimum of the average mutual information. The dynamics of the system is approximated using a map:

$$x_{i+1} = x_i + \Delta t \mathbf{W}_{\text{out}} \mathbf{O}(x_i, x_{i-n_\tau}, \ldots, x_{i-(k-1)n_\tau})$$

The feature vector $\mathbf{O}_i = \mathbf{O}(x_i, \ldots)$ is composed of biased, linear and nonlinear terms:

$$\mathbf{O}_i = \begin{bmatrix} 1 \\ \mathbf{O}_i^{(1)} \\ \mathbf{O}_i^{(2)} \\ \vdots \\ \mathbf{O}_i^{(m)} \end{bmatrix}$$

Linear term $\mathbf{O}_i^{(1)}$:

$$\begin{bmatrix} x_i \\ x_{i-n_\tau} \\ x_{i-2n_\tau} \\ \vdots \\ x_{i-(k-1)n_\tau} \end{bmatrix}$$

Nonlinear terms $\mathbf{O}_i^{(2)}$, $\mathbf{O}_i^{(3)}$, …:

$$\begin{bmatrix} x_i^2 \\ x_i x_{i-n_\tau} \\ x_i x_{i-2n_\tau} \\ \vdots \\ x_{i-n_\tau}^2 \\ \vdots \\ x_{i-(k-1)n_\tau}^2 \end{bmatrix} \quad \begin{bmatrix} x_i^3 \\ x_i x_{i-n_\tau} x_{i-2n_\tau} \\ x_i x_{i-n_\tau}^2 \\ x_i x_{i-2n_\tau}^2 \\ \vdots \\ \vdots \\ x_{i-(k-1)n_\tau}^3 \end{bmatrix} \quad \ldots$$

The linear term includes k points from the past signal. For NG-RC case the nonlinear term $\mathbf{O}^{(l)}$ is a set of all unique monomials of degree l composed of all components of the linear term $\mathbf{O}^{(1)}$. Nonlinear terms for the NG-RC-Ch case are obtained similarly, by replacing the monomials with Chebyshev polynomials of the corresponding degree, for example, $x^{j_1} y^{j_2} \to T_{j_1}(x) T_{j_2}(y)$.

Row vector $\mathbf{W}_{\text{out}}$ of output weights is determined in the training phase according to the requirement that the map provides the best approximation of the time series. By introducing the notations:

$$\mathbf{Y} = [x_n - x_{n-1}, x_{n-1} - x_{n-2}, \ldots] / \Delta t \quad \text{and} \quad \boldsymbol{O} = \begin{bmatrix} | & | & \\ \boldsymbol{O}_{n-1}, & \boldsymbol{O}_{n-2}, & \ldots \\ | & | & \end{bmatrix}$$

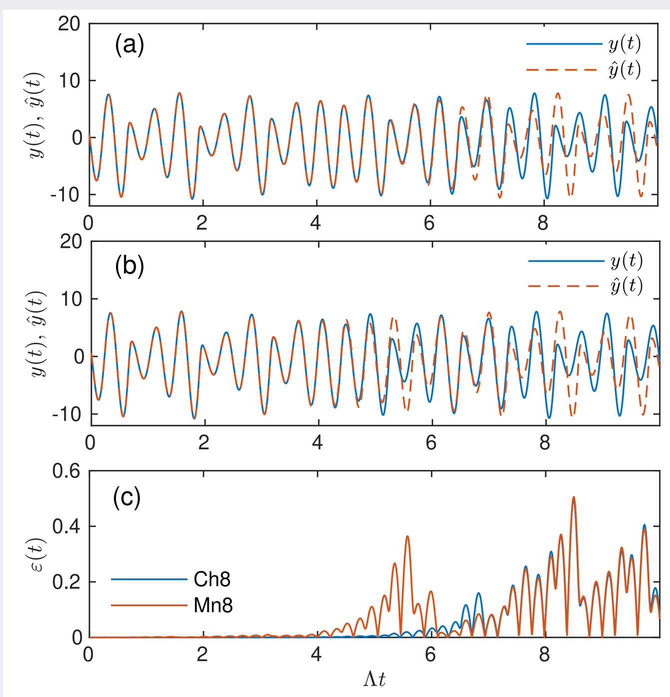
$\mathbf{W}_{\text{out}}$ is obtained by the least-squares method with Tikhonov regularization:

$$\mathbf{W}_{\text{out}} = \mathbf{Y}\mathbf{O}^T \left(\mathbf{O}\mathbf{O}^T + \beta \mathbf{I}\right)^{-1}$$

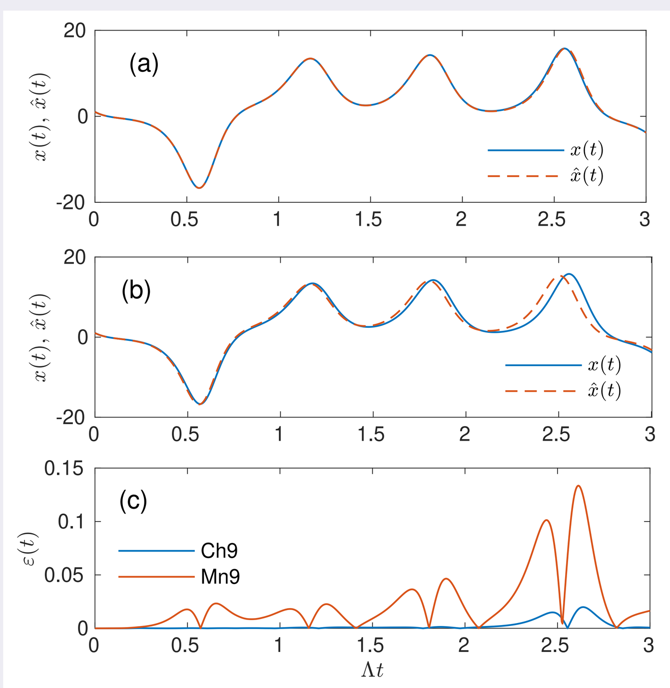
Here β is the regularization parameter, also known as ridge parameter, and $\mathbf{I}$ is the identity matrix.

Results

Rössler

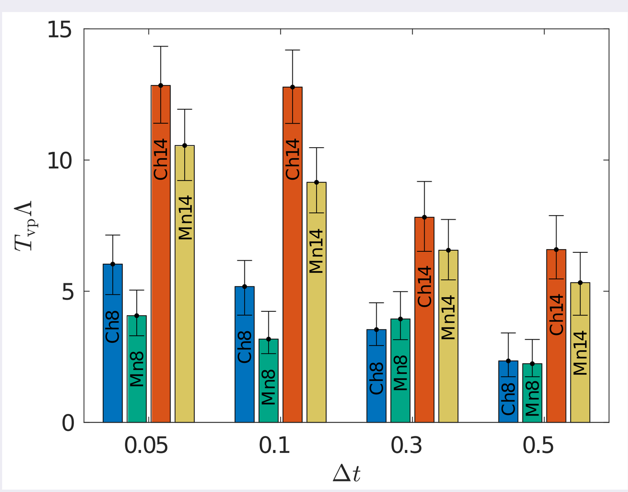


Lorenz

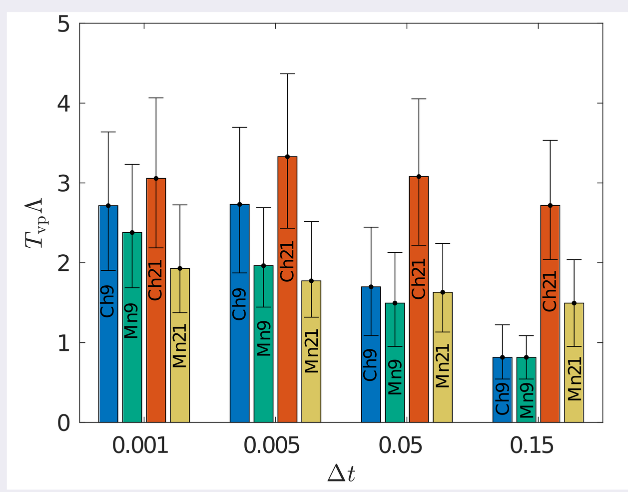


Examples of short-term prediction of the dynamics of Rössler and Lorenz systems by the NG-RC-Ch and NG-RC algorithms. The blue continuous curves in (a) and (b) show the original time series. The time series predicted by the NG-RC-Ch (a) and NG-RC (b) algorithms are shown by the red dashed curves. The blue and red curves in (c) show the normalized absolute errors for the NG-RC-Ch and NG-RC algorithms, respectively. The time axis is normalized by the largest Lyapunov exponent. The parameters for the Rössler system: $\Delta t = 0.1$, $T_{\text{train}} = 1000$, $k = 3$, $\tau = 1.5$ and $m = 8$ are the same for both algorithms; β is 10^{-7} for NG-RC-Ch and 10^{-3} for NG-RC. The parameters for the Lorenz system: $\Delta t = 0.005$, $T_{\text{train}} = 100$, $k = 3$, $\tau = 1.5$, $m = 9$ and $\beta = 10^{-9}$.

Rössler

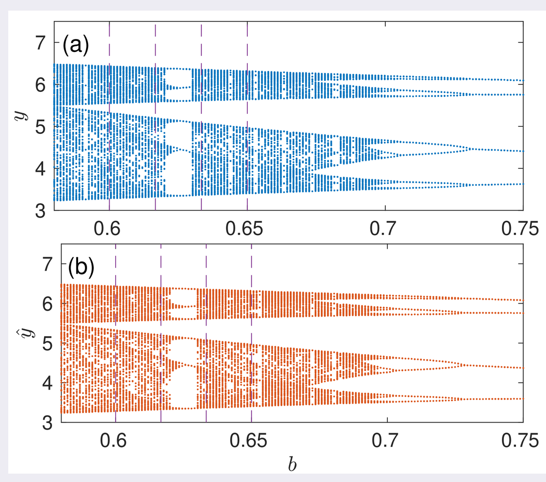


Lorenz

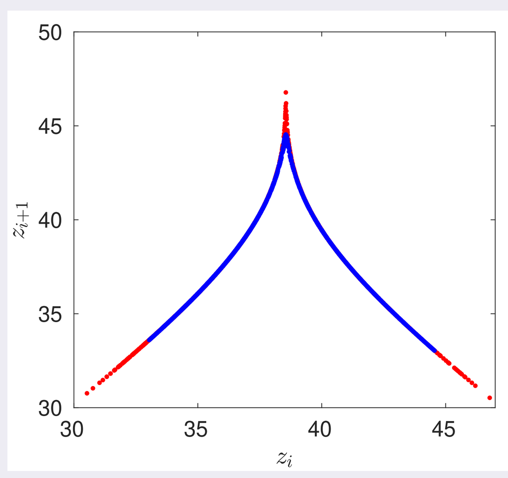


The medians of valid prediction time as a function of sampling time and degree of nonlinearity. The valid prediction time T_{vp} is defined as the time interval from the beginning of the prediction to the first point in time when the absolute value of the error exceeds some arbitrary threshold value. The prediction time is inhomogeneous on the attractor; the prediction result depends on the choice of the initial conditions. Here the valid prediction time is averaged over $N = 1000$ trajectories with different initial conditions on the strange attractor. Error bars show the first and third quartiles. We use the labels "Mn" (monomials) and "Ch" (Chebyshev polynomials) followed by the number m to denote the results of the NG-RC and NG-RC-Ch algorithms with a given degree m of nonlinearity, respectively. The parameters for the Rössler system: $T_{\text{train}} = 2000$, $k = 3$ and $\tau = 1.5$. The parameters for the Lorenz system: $T_{\text{train}} = 200$, $k = 3$ and $\tau = 0.15$.

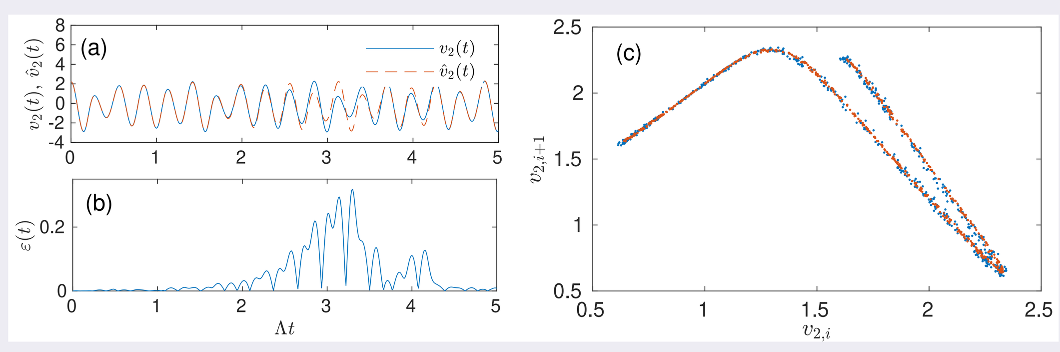
Rössler



Lorenz



Long-term (climate) prediction from a scalar observable: Reconstructing the bifurcation diagram of the Rössler system and the return map of the Lorenz system. (left) The dots show the local maxima of the variable $y(t)$ of the Rössler system obtained (a) from the original equations and (b) from the trained prediction model. The four vertical dashed lines indicate the values of the bifurcation parameter at which time series of the variable $y(t)$ were recorded for training the prediction model based on the NG-RC-Ch algorithm with $m = 7$ degree of nonlinearity. (right) Return map of the actual and predicted variable z of the Lorenz system. The blue dots refer to the actual Lorenz system and the red dots overlaying the blue dots refer to the prediction. The degree of nonlinearity of NG-RC-Ch algorithm is $m = 9$.



Application of NG-RC-Ch to experimental time series of the electronic Rössler-like oscillator. In real experiments noise perturbations, measurement errors, and other imperfections are inevitable. We verified the performance of our algorithm using experimental datasets of the electronic Rössler-like oscillator provided by Vera-Ávila et al [3]. (a) Dynamics of actual (blue curve) and predicted (red dashed curve) time series. (b) Dynamics of the prediction error. (c) The return maps constructed from the original (blue dots) and predicted (red dots) time series. The red dots are consistently near the blue dots, indicating that the trained model reproduces the climate in the long term. Note that the red dots fall on an imaginary thin continuous curve, while the blue dots are scattered (due to noise present in the experimental time series) in the neighborhood of this curve. Thus, it can be concluded that the NG-RC-Ch works as a noise filter.

Summary

- NG-RC and NG-RC-Ch with monomials or Chebyshev polynomials of sufficiently high degree can effectively predict the behavior of chaotic systems when only scalar time series is available for observations.
- The prediction time of NG-RC-Ch is approximately two Lyapunov time units longer than that of NG-RC.
- Our algorithm is suitable for short-term and long-term forecasting, including the reconstruction of Lyapunov exponents, return maps and bifurcation diagrams. It works well for time series recorded in a real experiment.

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References:
[1] D. J. Gauthier, I. Fischer, and A. Röhm, Chaos **32**, 113107 (2022).
[2] I. Ratas and K. Pyragas, Phys. Rev. E **109**, 064215 (2024).
[3] V. P. Vera-Ávila et al., Data in Brief **28**, 105012 (2020).